THROUGHPUT AND TIME DISTRIBUTION EQUATIONS REVISITED

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ABSTRACT. Time distribution (or throughput) equations are approximated often in the extremes of either high or low sample count rates. If the sample and background count rates are comparable, and if all terms are treated independently, dual-valued domains of count time can occur. These can be removed by accepting the constraint of total count time minimization and a good representation for $E = f(R_b)$.

INTRODUCTION

Previous authors (Loevinger & Berman 1951; Porges 1968; Putman 1962; Thomas 1950; Wyld 1970) have discussed the relation between E^2/R_b and T_a where E is counting efficiency, R_b is the count rate of the background and T_a is the count time of the analytical sample. Wyld (1970) illustrates the approach. (Table 1 provides definition for the terms used here. Table 2 summarizes the "error terms" and their interconversions.) Beginning with Equation (1), Wyld substituted Equation (2) for each term on the right side of Equation (1) and then substituted Equations (3A) and (3B) to obtain Equation (4). Equation (4) shows that the count time, T_a, of the analyzed sample at the acceptable statistical error, σ_{rd}, is determined by the sample count rate and efficiency in the form given.

$$\sigma_{rs}^2 = \sigma_{ra}^2 + \sigma_{rb}^2 \tag{1}$$

$$\sigma_{ri} = \left(\frac{R_i}{T_i}\right)^{1/2} \tag{2}$$

$$R_a = R_s + R_b \tag{3A}$$

$$R_d = \frac{R_s}{E} \tag{3B}$$

$$\sigma_{rd}^2 T_a = \frac{R_d}{E} + \frac{R_b}{E^2}\left(1 + \frac{T_a}{T_b}\right) \tag{4}$$

Two useful working approximations have been made in terms of the ratio, R_s/R_b. If $T_b \gg T_a$ and $R_a \gg R_b$, meaning the sample count rate is much larger than the background count rate, then Equation (4) becomes

$$T_a E = \frac{R_d}{\sigma_{rd}^2} \ . \tag{5}$$

TABLE 1. Definition of Symbols

Symbol	Definition
B	Background counts
S	Sample counts (B not included)
R_b	Count rate of the background
R_s	Count rate of the net sample
R_a	Count rate of the analytical sample ($= R_b + R_s$)
T_b	Count time of the background
T_s	Count time of the net sample
T_a	Count time of the analytical sample (numerically $= T_s$)
D	Actual disintegration counts
R_d	Actual disintegration count rate
E	Fractional counting efficiency ($= R_s/R_d$)

TABLE 2. "Error Terms", σ_i, as a Function of Counts, Count Rate and % Relative Error*

General	A function of	Example
$\sigma_c = C^{1/2} = (RT)^{1/2}$	Counts	$\sigma_b = B^{1/2} = (R_bT_b)^{1/2}$
$\sigma_r = \sigma_c/T = (R/T)^{1/2}$	Count rate	$\sigma_{rs} = (\sigma/T)_s = (R/T)_s^{1/2}$
$\sigma_{k\%} = 100kC^{1/2}/C = 100k/\sigma_c$	% relative error	$\sigma_{2a\%} = 200/\sigma_a$

*For interconversion among the three error terms, $\sigma_c = \sigma_r T = 100\ k/\sigma_{k\%}$, where k is the constant determined by the confidence level desired.

If $T_b \gg T_a$ and $R_b \gg R_a$, meaning the background count rate is much larger than the sample count rate, then Equation (4) becomes

$$\frac{T_a E^2}{R_b} = \frac{1}{\sigma_{rd}^2} \quad . \tag{6}$$

Equation (5) points out that T_aE is a constant for a high sample count rate and given error condition. Equation (6) notes that $T_a(E^2/R_b)$ is a constant for a low sample count rate and given error condition.

Alternatively, if two counting conditions are being compared for selection of the most favorable of two count times, then Equation (5) provides

$$T_{a1}E_1 = T_{a2}E_2 \quad . \tag{7}$$

for high sample count rates. The smallest count time, T_{a1}, requires the largest counting efficiency, E_1. Analogously for low sample count rates, Equation (6) selects between the most favorable of two count times in terms of E^2/R_b. The smallest count time, T_{a1}, requires the largest $(E^2/R_b)_1$

$$\frac{T_{a1}E_1^2}{R_{b1}} = \frac{T_{a2}E_2^2}{R_{b2}} \quad . \tag{8}$$

Discussion

Equations (4–8) characterize the usual conclusions drawn in the context of count time minimization or sample throughput discussions. The above conclusions are based on minimizing T_a, not the total count time, T, which equals $T_a + T_b$. If the terms in Equation (4) are treated independently and the full equation is used rather than approximations, dual-valued domains occur. Figure 1 illustrates one such example. Here, a reduction in count time occurs with an increase in E^2/R_b in one region whereas both increase in another region. This result depends on $T_b \gg T_a$ and an assumed function for $E = f(R_b)$. Consider deriving Equation (4) from the viewpoint of minimization of total count time. This will clarify a constraint on the ratio of analytical sample to background count times, (T_a/T_b).

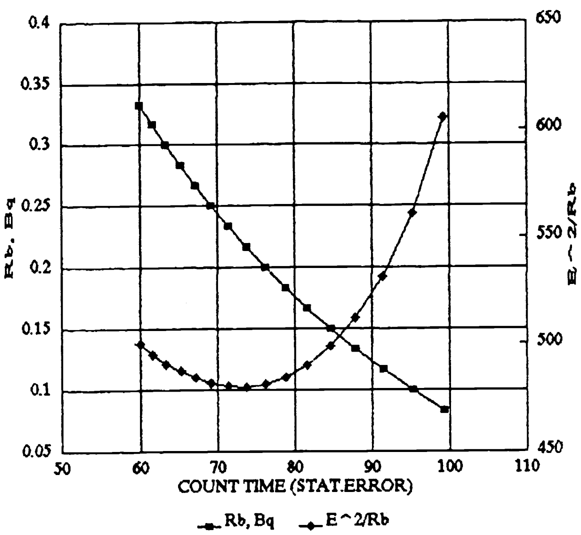

Fig. 1. R_b and E^2/R_b in terms of count time; R_s = 0.33, R_b = 0.33 → 0.08, E = 1 → 0.55, T_a/T_b = 1

Total Count Time Minimization

The total count time, T, for N_b background samples and N_a analytical samples is given by

$$T = N_b T_b = N_a T_a \tag{9}$$

if T_b and T_a are the same for all N_b and N_a, respectively.

Because N_a and N_b are constant in a given experiment, the minimization equation is

$$dT = N_b dT_b = N_a dT_a = 0$$

so that

$$N_b dT_b = -N_a dT_a \quad . \tag{10}$$

If the time differentials are obtained from the appropriate "statistical error" equations, substituted into Equation (10) and developed, then the results obtained are as shown in Table 3. For convenience, Table 3 gives results using "error terms" of 2 σ%, percent relative error (the form found on liquid scintillation counters) and σ_r, the form of count rate (Wyld 1970).

TABLE 3. Minimum Count Time Equations in Terms of $\sigma_{2\%}$ and σ_{rd}

Count time	$\sigma_{2\%}$ form	σ_{rd} form for $N_b = N_a = 1$
$T_b =$	$4\times10^4 \dfrac{(R_b)^{1/2} f(R_b,R_a)}{(N_b)^{1/2}\sigma^2_{\%s}R_s^2}$ *	$\dfrac{R_b\left(1 + \dfrac{T_a}{T_b}\right)}{\sigma^2_{rd}E^2}$
$T_a =$	$4\times10^4 \dfrac{(R_a)^{1/2} f(R_b,R_a)}{(N_a)^{1/2}\sigma^2_{\%s}R_s^2}$ *	$\dfrac{R_s + R_b\left(1 + \dfrac{T_a}{T_b}\right)}{\sigma^2_{rd}E^2}$
$T =$	$\dfrac{4\times10^4 + f(R_b,R_a)^2}{\sigma^2_{\%s}R_s^2}$ *	$\dfrac{R_s + 2R_b\left(1 + \dfrac{T_a}{T_b}\right)}{\sigma^2_{rd}E^2}$

* $f(R_b,R_a) = (N_bR_b)^{1/2} + (N_aR_a)^{1/2}$

From Table 3, the ratio of analytical to background count times is equal to the square root of the ratio of the analytical to background count rates

$$\frac{T_a}{T_b} = \frac{\left(\dfrac{R_a}{R_b}\right)^{1/2}}{\left(\dfrac{N_a}{N_b}\right)^{1/2}} = \left(\frac{R_a}{R_b}\right)^{1/2}$$

where $N_a = N_b$ for simplicity. Substitution of the identity

$$E^2\sigma^2_{rd} = \frac{R_s^2\sigma_{\%s}}{4\times10^4} \qquad (11)$$

into T_a gives

$$T_a\sigma^2_{rd} = \frac{R_d}{E} + \frac{R_b}{E^2}\left[1 + \left(\frac{R_a}{R_b}\right)^{1/2}\right] \qquad (12)$$

which becomes Equation (4) after the substitution of

$$\left(\frac{R_a}{R_b}\right)^{1/2} = \frac{T_a}{T_b} \ .$$

RESULTS

Equation (4) derived from statistical origins and Equation (12) derived from minimization of total count time have been shown equivalent. This means that T_a/T_b must equal $(R_a/R_b)^{1/2}$ to achieve minimization of total count time. Approximations that assume $T_a/T_b = 1$ or $= 0$ do not provide minimization of total count time. Also, a realistic $E^2/R_b = f(R_b)$ is required for an experimentally verifiable result. No single function will be applicable because a variety of experimental conditions is possible. But to illustrate, a function was obtained from a liquid scintillation counter for one pulse-height range and is

$$E^2 = (689.5 - 13.9R_b)R_b \ . \qquad (13)$$

Figure 2 is a plot of T_a, T_b and T in terms of E^2/R_b and ignores the constraint just discussed. T_a and T_b have been treated as independent terms. As previously noted, T_a and T_b are dual-valued functions increasing in one domain with E^2/R_b and decreasing in another. Figure 3 plots the same parameters with Equation 13 included and $T_a/T_b = (R_a/R_b)^{1/2}$. Now count times are single-valued and decrease in the expected fashion as E^2/R_b increases. (Note that Table 3 provides the count time distribution between T_a and T_b for any number of analytical, N_a, and background, N_b, samples.)

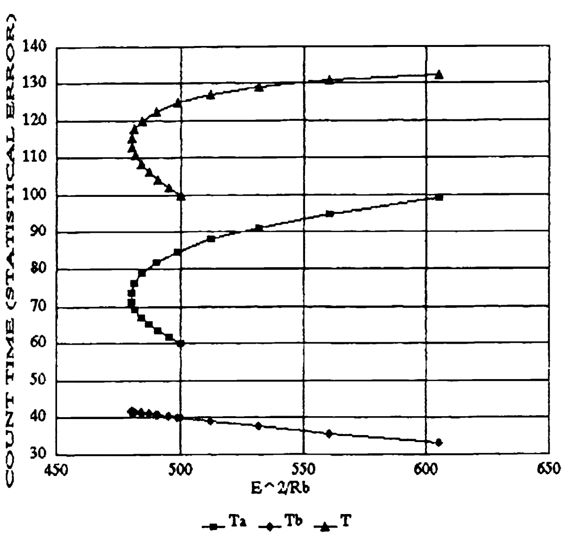

Fig. 2. Count times: T_a, T_b and T as a function of E^2/R_b, the pairs T_a:T_b and E:R_b treated independently

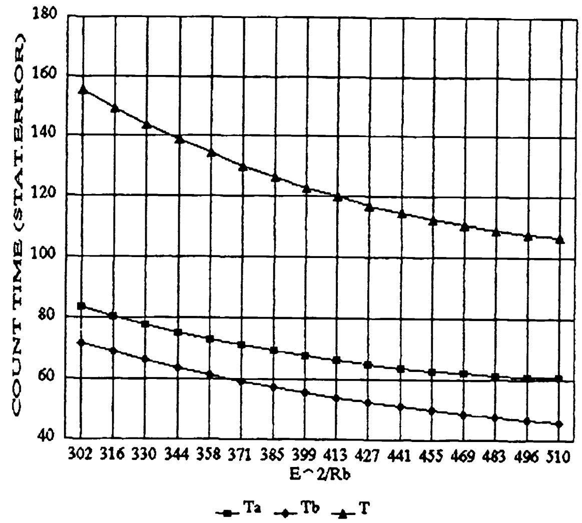

Fig. 3. Count times: T_a, T_b and T as a function of E^2/R_b; E is a measured function of R_b and $(T_a/T_b) = (R_a/R_b)^{1/2}$

CONCLUSION

Approximations made to the statistically derived count time equation for T_a at high and low sample count rates relative to background give good results. In the region of comparable background and sample count rates, the complete equation with all terms treated independently can produce domains of dual-valued count times. This is relieved by accepting the constraint of total count time minimization and an accurate representation for $E^2/R_b = f(R_b)$. Minimization of total count time means that $T_a/T_b = (R_a/R_b)^{1/2}$.

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